12^{th} Annual Johns Hopkins Math Tournament
Saturday, February 19, 2011

General Test 1

1. [**1025**] Let $F(x)$ be a real-valued function defined for all real $x \neq 0, 1$ such that
$$F(x) + F\left(\frac{x-1}{x}\right) = 1 + x.$$
Find $F(2)$.

2. [**1026**] Find the number of pairs (a, b) with a, b positive integers such that $\frac{a}{b}$ is in lowest terms and $a + b \leq 10$.

3. [**1028**] Find all rational roots of $|x-1|\left|x^2-2\right| - 2 = 0$.

4. [**1032**] Let $M = (-1, 2)$ and $N = (1, 4)$ be two points in the plane, and let P be a point moving along the x-axis. When $\angle MPN$ takes on its maximum value, what is the x-coordinate of P?

5. [**1040**] Mordecai is standing in front of a 100-story building with two identical glass orbs. He wishes to know the highest floor from which he can drop an orb without it breaking. What is the minimum number of drops Mordecai can make such that he knows for certain which floor is the highest possible?

6. [**1056**] Let ABC be any triangle, and D, E, F be points on BC, CA, AB such that $CD = 2BD$, $AE = 2CE$ and $BF = 2AF$. Also, AD and BE intersect at X, BE and CF intersect at Y, and CF and AD intersect at Z. Find the ratio of the areas of $\triangle ABC$ and $\triangle XYZ$.

7. [**1088**] Find the projection of the sphere $x^2 + y^2 + (z-1)^2 = 1$ onto the plane $z = 0$ with respect to the point $P = (0, -1, 2)$. [The answer will be some conic curve. Express the equation in the form $f(x, y) = 0$ where f is some quadratic in x and y.]

8. [**1152**] It is a well-known fact that the sum of the first n kth powers can be represented as polynomials in n. Let $P_k(n)$ be such polynomial. For example, one has $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, so one has $P_2(x) = \frac{x(x+1)(2x+1)}{6} = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x$. Evaluate $P_3(-4) + P_4(-3)$.

9. [**1280**] Find the final non-zero digit in 100!. For example, the final non-zero digit of 7200 is 2.

10. [**1536**] How many polynomials P of degree 4 satisfy $P(x^2) = P(x)P(-x)$?